APPLICATION OF ASYMPTOTIC METHODS OF THE THEORY OF NONLINEAR OSCILLATIONS TO WAVE PROPAGATION IN AN INHOMOGENEOUS MEDIUM

I. F. Budagyan and D. I. Mirovitskii

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Application of the method given in [1] for solving the wave propagation problem in inhomogeneous media with nonmonotonic and oscillating dependence of the wave number on position is discussed.

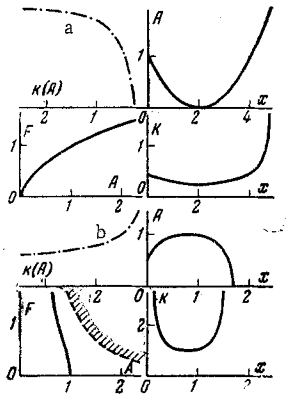

Fig. 1

The method described earlier in [1] involves a transformation from the equation $\psi'' + k^2\psi = 0$ for the resultant field $\psi = \alpha + \beta$ in an inhomogeneous medium with $k = k(x)$ to equations written in terms of the amplitude and phase of the direct partial wave $\alpha(x) = A(x)\, e^{i\varphi(x)}$ with subsequent change of variable $x \to A$. The application of the asymptotic method of the theory of nonlinear oscillations [2, 3], which reduces to the determination of phase trajectories on the phase plane, then enables us to solve the resulting differential equation. The change of variable $A \to x$ is performed at the end. The direct (α) and reverse (β) partial waves, the resultant field ψ, and the function $k = k(x)$ are determined by the form of the chosen function $k = k(A)$. This method enables us to consider media in which the wave number increases or decreases monotonically with distance, varies nonmonotonically, or oscillates with constant or variable period and amplitude, i.e., its range of validity is not restricted to any definite class of inhomogeneous media, and its accuracy depends only on the accuracy of the corresponding graphical constructions or numerical calculations.

§1. TRANSFORMATION FROM GIVEN LAW k = k(x) TO k = k(A)

The direct problem of the theory of wave propagation can be reduced to the determination of the field ψ in an inhomogeneous medium $k = k(x)$. The choice of the function $k = k(A)$ is quite readily carried out on the basis of the function $k = k(x)$ characterizing a given inhomogeneous medium. In fact, consider the main relations which are necessary for the solution of the problem of wave propagation, and the resulting functional relations between $k(x)$ and $A(x)$, $A(x)$ and $F(A)$, and $F(A)$ and $k(A)$. It will be shown that these relations can be used to find the particular function $k = k(A)$ which leads to the given law $k = k(x)$.

(a) There are two possible relationships between the given law $k = k(x)$ and the auxiliary function $A = A(x)$ that are determined by the form of the function $k = k(A)$. This is clearly shown by, for example, Fig. 1. When the character of the function $A = A(x)$ is the same as that of $k = k(x)$, the function $k(A)$ increases monotonically along the OA axis. If the over-all behavior of $A = A(x)$ is opposite to that of the given function $k = k(x)$, the function $k(A)$ decreases monotonically along the OA axis, and the function $A = A(x)$ has an arbitrary shape, as shown in Fig. 1a, then the values $A = 1, 0, 1$ correspond to $x = 0, 2, 4$. These values of A, in turn, correspond to wave numbers $k = 0.45, 0.25, 0.45$. Thus, the law $k = k(x)$ corresponds to the function $A = A(x)$ in its over-all variation along the x-axis.

When the function $k(A)$ decreases monotonically along the OA axis, we have the opposite situation shown in Fig. 1b. Here, the minima of the $k = k(x)$ curve correspond to the maxima of the $A = A(x)$ curve, and regions where the $k = k(x)$ curve falls (or rises) correspond to the rise (or fall) of the function $A = A(x)$. Consequently, in this case, the over-all character of the function $k = k(x)$ is opposite to that of $A = A(x)$.

For a nonmonotonic function $k(A)$, for example, in the case of a periodic form of this function, the phase plane splits into a number of independent regions, each having its own solution corresponding to a particular wave number variation $k = k(x)$. In each such independent region the function $k(A)$ varies monotonically and, consequently, this case reduces to the first or second variant of the relationship between $k = k(x)$ and $A = A(x)$ which we have just discussed, depending on the character of the function $k(A)$.

(b) The relation between $A = A(x)$ and $F = F(A)$ follows from the definition of the function $F = dA/dx$ given in [1]. Transformation to the curve $F = F(A)$ is achieved by differentiating $A(x)$, i.e., by finding dA/dx and introducing the function $x = x(A)$ obtained from the curve $A = A(x)$. Thus, for the symmetric layer which is widely used in the theory of radiowave propagation (in estimating the effect of the ionosphere on radio communications) and in quantum mechanics (tunnel effect), we have

$$A(x) = N - 4Me^{\gamma(x-c)}[1 + e^{\gamma(x-c)}]^{-2}, \tag{1.1}$$

for which

$$\frac{dA}{dx} = -4M\gamma e^{\gamma(x-c)}\,[1 - e^{\gamma(x-c)}]\,[1 + e^{\gamma(x-c)}]^{-3}, \tag{1.2}$$

and if we determine the function $e^{\gamma(x-c)}$ from Eq. (1.1), and substitute it into Eq. (1.2), we can readily show that

$$F(A) = \pm\gamma\,(N - A)\,[M^{-1}(M - N + A)]^{1/2}. \tag{1.3}$$

This leads to the symmetric (relative to the OA axis) curve given by Eq. (1.3), which cuts the OA axis at the points $A = N - M$ and $A = N$; at the first of these points the tangent is parallel to the OF axis (see Figs. 2, 1).

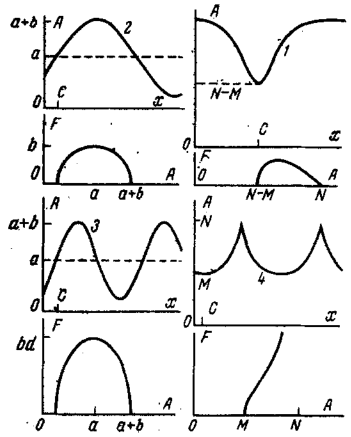

Fig. 2

Similarly, we can find the function $F = F(A)$ for $A = A(x)$ given by the following formulas:

$$A(x) = a + b\sin(x - c), \qquad F^2 = b^2 - (A - a)^2 \tag{1.4}$$

$$A(x) = a + b \sin d (x - c),$$
$$F^2 = (bd)^2 [1 - b^{-2}(A - a)^2], \qquad (1.5)$$

$$\operatorname{arc\,tg} [(N - A)(A - M)^{-1}]^{1/2} =$$
$$= [(A - M)(N - A)]^{1/2} - (x - c),$$
$$F^2 = (A - M)(N - A)^{-1}; \qquad (1.6)$$

$$A(x) = ax + b, \qquad F = a, \qquad (1.7)$$

$$A(x) = (ax + b)^{1/2}, \qquad F = {}^1\!/_2\, aA^{-1}, \qquad (1.8)$$

$$A(x) = \ln (ax + b), \qquad F = a \exp(-A), \qquad (1.9)$$

$$A(x) = a(x - c)^2 + b, \qquad F^2 = 4a(A - b). \qquad (1.10)$$

Hence it follows that for the sinusoidal function of Eq. (1.4), plotted on the Ax plane, we obtain on the phase plane FA a circle of radius b, which is located symmetrically relative to the OA axis with the center at a distance a from the origin. The sinusoidal function given by Eq. (1.5) with the variable period $(x - c) = 2\pi d^{-1}$, on the other hand, produces an ellipse in the FA plane. The inverted cycloid on the Ax plane corresponds to the curve of Eq. (1.6) on the phase plane with a vertical asymptote at the point A = N, which is symmetric relative to the OA axis and cuts it vertically at A = M. The straight line Eq. (1.7) corresponds on the FA plane to a straight line parallel to the OA axis. The function given by Eq. (1.8), which was discussed in [1], leads to an exponential, and the parabola of Eq. (1.10) leads to a parabola which is symmetric relative to the OA axis and cuts it at the point A = b. These results are illustrated in Figs. 2 and 3.

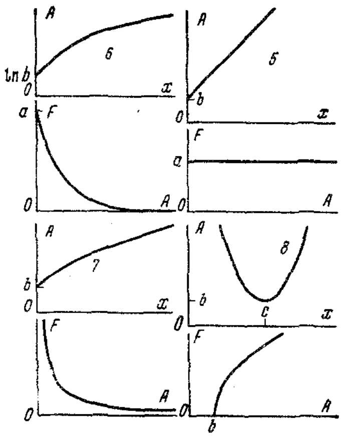

Fig. 3

(c) The relation between the the functions F = F(A) and k = k(A) is established by the differential Eq. (2.5) in [1] and, therefore, for any function k = k(A) we can readily construct the corresponding curves F = F(A) on the phase plane (all the formulas in [1] to which we shall refer will henceforth be indicated by an asterisk to distinguish them from the formulas in the present paper). The use of the equation of the limiting trajectory, given by Eq. (3.2*), will help us to establish the character of the function k = k(x), since the shape of the phase trajectories, i.e., the curves F = F(A), is in a certain definite correspondence with the shape of the limiting trajectory, and the phase trajectories cut the OA axis vertically in the phase plane with the exception of singular points at which $f_1 = k(A) + 2k(A)A^{-1} = 0$. The case of several singular points on the phase plane, or their occupation of the entire OA axis, is encountered only in certain special problems. To construct the curves F = F(A) in the neighborhoods of singular points, we must carry out an additional analysis or use the complete Eq. (2.5*) instead of Eq. (3.2*).

As an example, consider the function $k(A) = A^{-m}$. In accordance with Eqs. (2.1*), (2.6*), (2.7*), and (2.8*) we have

$$p(A) = \frac{1 - m}{A}, \quad f_1 = \frac{2 - m}{A^{m-1}}, \quad f_2 = \frac{(2 - m)(1 - m)}{A^{3-m}},$$

$$\Phi = \frac{1}{A^m} - \left[\frac{1 - m}{A^2} F^2 + \frac{1}{A^{2m}}\right]^{1/2}.$$

Equation (2.5*) will therefore assume the form

$$\frac{dF}{dA} = \frac{F}{A}\left[(1 - m) + \frac{2 - m}{1 - G^{3/2}} G\right],$$
$$G = 1 + \frac{(1 - m)F^2}{A^{2(1-m)}}, \qquad (1.11)$$

and the equation for the limiting trajectory (3.2*) will be

$$F = A^{1-m}(m - 1)^{-1/2}.$$

Singular points occur at m = 2 (in this case, $f_1 = 0$ along the entire OA axis). This case is considered in detail in [1], where it is shown

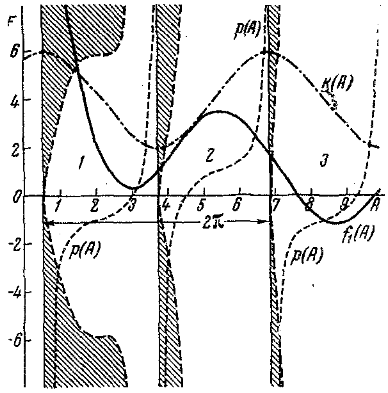

Fig. 4

that the limiting trajectory on the phase plane is an equilateral hyperbola. In the remaining cases, i.e., for m > 2, there are no singular points since f_1 does not vanish for any A. Integration of Eq. (1.11) gives the following equation for the phase trajectories:

$$F^2(A) = \frac{aA^{4(1-m)}(2A^m - aA^2)}{(1 - m)(aA^{2-m} - 1)^2}. \qquad (1.12)$$

In particular, for m = 3 we can write

$$F(A) = \frac{dA}{dx} = \pm\left[a\left(\frac{a}{2} - A\right)\right]^{1/2} \frac{1}{(a - A)A^2}$$
$$\left(0 \leqslant A \leqslant \frac{a}{2}\right). \qquad (1.13)$$

In contrast to phase trajectories, the limiting trajectories retain their form between m = 2 and m = 4. For m > 2 the phase trajectories cut the OA axis vertically since $f_1 \neq 0$ [see the F = F(A) curve based on Eq. (1.13) and given in Fig. 1b]. The corresponding function A = A(x) can be found by writting Eq. (1.13) in the form

$$\pm A^2 (a - A)\, dA = [a(^1\!/_2 a - A)]^{1/2} dx,$$

and integrating

$$x - c = \pm 2 \frac{g}{\sqrt{a}}\left[\frac{1}{7} g^3 - \frac{1}{10} ag^2 - \frac{1}{3}\left(\frac{a}{2}\right)^2 g + \left(\frac{a}{2}\right)^3\right],$$

where c is the displacement of the curve A = A(x) along the x axis, and $g = (a/2) - A$.

If the differential

$$kk'' - 2(k')^2 - AF^{-2}(A)\, k^3 k' = 0, \qquad (1.14)$$

obtained from Eq. (3.2*) for the limiting trajectory with allowance for (2.1*), has an exact solution, then the function k = k(A) corresponding to the phase trajectory F = F(A) is immediately determined. In fact, for case (b) of Section 1 we find for Eq. (1.10) that $F^2 = 4aA$ when b = 0, and Eq. (1.14) assumes the form

$$kk'' - 2(k')^2 - \frac{k^3}{4a} k' = 0. \qquad (1.15)$$

The solution of (1.15) gives an equation which determines the function $k = k(A)$, i. e.,

$$(4ab)^{-1} \ln [bk^{-1}(A) + (4a)^{-1}] - k^{-1}(A) = B(A - D)$$

where B and D are integration constants.

Figure 1a shows a plot of $k = k(A)$ corresponding to the chosen function (1.10) for $A = A(x)$, and for the improved law $k = k(x)$. It is assumed in this plot that $a = 1/4$, $c = 2$, $D = 2$, and $B = 1$.

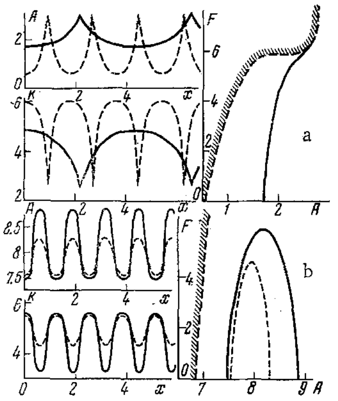

Fig. 5

Thus, to determine the function $k = k(A)$ corresponding to $F = F(A)$ of the necessary form, we must use the equation for the limiting trajectory, integrate it (if possible), or use it to construct the "estimated" limiting trajectories. The phase trajectories approximately repeat the form of the limiting trajectories and cut the OA axis vertically everywhere except for the singular points.

§2. MAIN RESULTS OF ANALYSIS OF MORE COMPLICATED CASES

To solve the problem of wave propagation in a medium with an arbitrary $k = k(x)$ we must first determine the corresponding form of the phase trajectory and choose $k = k(A)$ so that it ensures that this particular trajectory is obtained. The method of [1] is then used to obtain by a graphical construction (or numerical calculation) the precise form of $k = k(x)$. Since in practical applications there is usually no need to follow successively all the various constructions leading to the "best" form of $k = k(x)$, we find at this stage the actual solution, i. e., the functions $\psi = \psi(x)$, $\alpha = \alpha(x)$, and $\beta = \beta(x)$ for the "best" form of $k = k(x)$.

In the most complicated case of propagation, i. e., an oscillating function $k = k(x)$, let us consider case b of Section 1 and, in particular, Eqs. (1.4), (1.5), and (1.6). It follows from Fig. 2 that the phase trajectories corresponding to $k = k(x)$ and $A = A(x)$ in the form of an inverted cycloid have a vertical asymptote for $F \to \infty$ and cut the OA axis vertically. Sinusoidal functions $k = k(x)$ and $A = A(x)$ correspond to closed phase trajectories of elliptical form, located symmetrically relative to the OA axis and shifted by an amount a from the origin along the OA axis. The shift a is equal to the displacement of the $A = A(x)$ curve on the Ax plane in the vertical direction (above the x axis). The semiaxis of the ellipse lying on the phase plane along the OA axis characterizes the amplitude of the oscillations in $A = A(x)$, while the semiaxis parallel to the OF characterizes the periods of the oscillations of $A = A(x)$ and $k = k(x)$. The shift of the oscillations of $A = A(x)$ along the x-axis, i. e., the quantity c, is completely determined by the boundary conditions, and is entirely arbitrary for a given phase trajectory, since it is a constant of integration during transformation from the phase plane to the Ax plane.

Closed phase trajectories can exist only near singular points of f_1 since the condition $dF/dA = \infty$ is not satisfied for these points. Therefore, closed phase trajectories (for example, elliptical trajectories) on the phase plane demand the pressure of singular points, but although this condition is necessary it is not sufficient. It follows that, in the case of an oscillating wave number, we must choose the function $k = k(A)$ so that it ensures the presence of at least one singular point, and then confine our attention to the neighborhood of this point.

In particular, consider the case

$$k(A) = b + \sin(A + a). \tag{2.1}$$

According to Eq. (2.1*) we then have

$$p(A) = \frac{2}{k}\frac{dk}{dA} - \left(\frac{dk}{dA}\right)^{-1}\frac{d^2k}{dA^2} =$$

$$= \frac{2 + \sin(A + a)[b - \sin(A + a)]}{[b + \sin(A + a)]\cos(A + a)}. \tag{2.2}$$

Substituting Eq. (2.2) into Eq. (3.2*) for the limiting trajectory, we have

$$F(A) = k\left(-\frac{A}{p}\right)^{1/2} =$$

$$= \frac{\{[\sin(A + a) + b]^3 A \cos(A + a)\}^{1/2}}{\{[\sin(A + a) - b]\sin(A + a) - 2\}^{1/2}}.$$

The directional field constructed on the phase plane AF splits into a number of regions, each of which is characterized by its own limiting trajectory and can be analyzed independently of the remaining regions.

Let us now specify the form of Eq. (2.1), i. e., choose, for example,

$$k(A) = 2[2 + \sin(A + 1)], \tag{2.3}$$

and denote by the hatched areas those regions of the phase plane of Fig. 4 which correspond to nonpropagating waves. Between these hatched regions we have regions corresponding to propagating waves, bounded by the limiting trajectories. The original function $k = k(A)$ is indicated by the dot-dash curve, and the curves $p = p(A)$ and $f_1 = f_1(A)$ are shown by the dashed and solid lines, respectively.

As an example, let us analyze the problem for regions 1 and 3 on the phase plane. Region 1 extends from the OA axis up to $A_{11} = 0.57$, where $k = k(A)$ has a maximum, and up to $A_{12} = 3$, where the $p = p(A)$ curve cuts the horizontal axis ($p = 0$). Region 3 extends from $A_{31} = 6.85$, which is shifted along the OA axis relative to A_{11} by 2π, and $a_{32} = 9.3$, where, again $p = 0$.

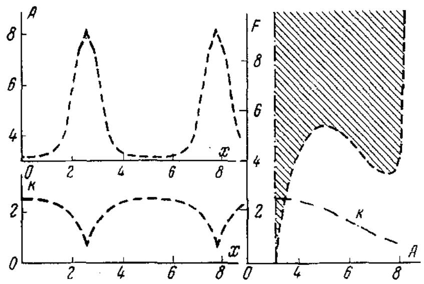

Fig. 6

1. Figure 5a shows region 1 of the phase plane of Fig. 4. Let us plot on this figure one of the phase trajectories $F = F(A)$ and construct on the left the functions $A = A(x)$ and $k = k(x)$ (these are shown by the solid lines). The dashed lines show the functions corresponding to the limiting trajectory which is indicated by the dashed curve. We emphasize that all these constructions are given for arbitrary boundary conditions, and that the shift of the curve $k = k(x)$ through an arbitrary amount along the x axis involves the same shift for the $A = A(x)$ curve.

2. The right-hand side of Fig. 5b shows region 3 of the phase plane of Fig. 4. If we restrict our attention to only two phase trajectories in this figure, it is sufficient to show only their upper halves because of the symmetry of the phase trajectories relative to the OA axis noted above. The centers of the trajectories shown by the solid and broken lines coincide with the singular point $A = 7.65$, but the trajectories are shifted somewhat along the OA axis and their shape is nearly elliptical. As before (region 1), the function $A = A(x)$ and $k = k(x)$ for both phase trajectories are shown on separate graphs on the left of Fig. 5b.

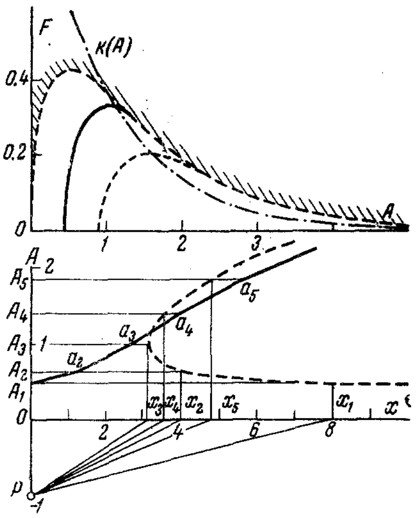

Fig. 7

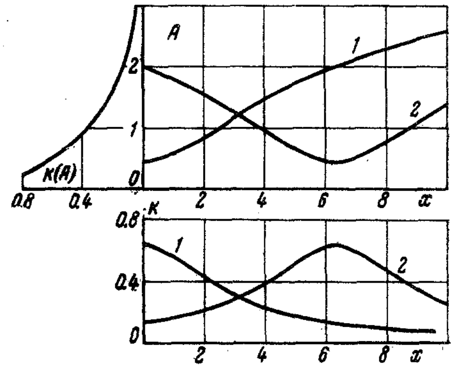

Fig. 8

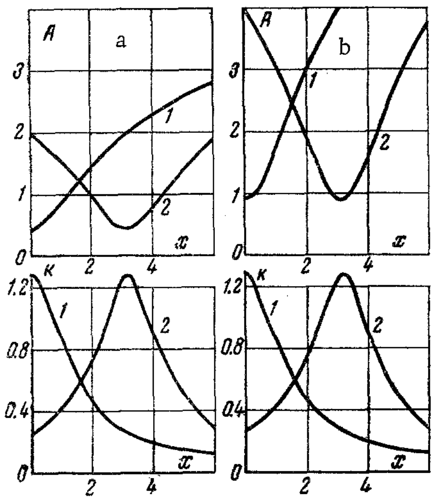

Fig. 9

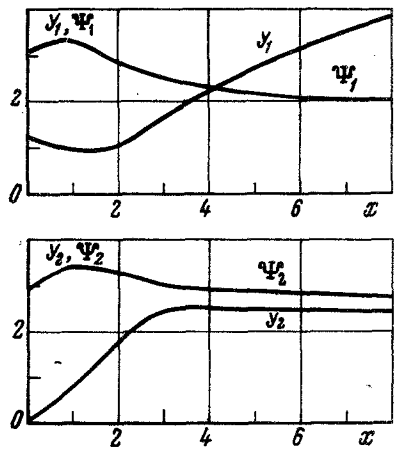

Fig. 10

Figure 6 shows the analogous constructions for the other special case of Eq. (2.1), namely,

$$k(A) = 1.5 + \sin 0.5A. \tag{2.4}$$

The limiting trajectory is shown by the dashed curve on the phase plane on the right-hand side of the figure, while the functions $A = A(x)$ and $k = k(x)$ are shown on the left. The function $k = k(A)$ is shown by the dot-dash curve.

It is clear from the above examples that a given function $k = k(A)$ can ensure the solution of the wave-propagation problem for a number of different types of inhomogeneous media. The function $k = k(A)$ must therefore be regarded as the basic function for this asymptotic method.

§3. EXAMPLE OF THE SOLUTION OF THE WAVE-PROPAGATION PROBLEM

Consider a medium characterized by the basic function

$$k(A) = \exp(-A) \tag{3.1}$$

shown in Fig. 7 by the dot-dash curve. As in case (c) of §1, to determine the phase and limiting trajectories from Eqs. (2.5*) and (3.2*), we must first use Eqs. (2.1*), (2.6*), (2.7*), and (2.8*) to determine the functions

$$p = -1, \quad f_1 = e^{-A}\left(\frac{2}{A} - 1\right), \qquad f_2 = \frac{e^A}{A}\left(1 - \frac{3}{2A}\right),$$
$$\Phi = e^{-A}\left[1 - \left(1 - \frac{e^{2A}F^2}{A}\right)^{1/2}\right]. \tag{3.2}$$

Using Eqs. (3.2) and (2.5*) for the angle θ between the tangent to the $F = F(A)$ curve and the horizontal axis at any point on the phase plane, we obtain

$$\operatorname{tg}\theta = F\left\{-1 + \left[\frac{2}{A} - 1 + \frac{1}{A}\left(1 - \frac{3}{2A}\right)e^{2A}F^2\right]\times\right.$$
$$\left.\times\left[1 - \left(1 - \frac{1}{A}e^{2A}F^2\right)^{1/2}\right]^{-1}\right\}, \tag{3.3}$$

and the equation for the limiting trajectory becomes

$$F(A) = k(-pA^{-1})^{-1/2} = A^{1/2}e^{-A}. \tag{3.4}$$

It follows from Eq. (3.4) that the limiting trajectory has a horizontal tangent at the point $A = 0.5$ on the phase plane. It is shown in Fig. 7 by the dashed curve. The region corresponding to nonpropagating waves is hatched. In accordance with (3.3), all the phase trajectories have horizontal tangents at phase-plane points determined by

$$F^2 = \frac{A}{(1 - {}^3/_2 A^{-1})e^{2A}}\left\{2\left(1 - \frac{1}{A}\right) - \right.$$
$$\left. - \frac{1 - [8A^{-1} - 3(1 + A^2)]^{1/2}}{2(1 - {}^3/_2 A^{-1})}\right\}. \tag{3.5}$$

Figure 7 shows the family of phase trajectories obtained from Eqs. (3.3) and (3.5) in the unhatched region of the phase plane.

To be specific, let us select one of the phase trajectories, for example, the one shown by the solid line, and determine the functions $A = A(x)$, $k = k(x)$, and the field $\psi = \psi(x)$. Graphical constructions involve the transfer (in two stages) of the curve from the phase plane to the Ax plane of Fig. 7. First, the curve is transformed from the AF set of coordinates to the Ax' set, where the curve $F = F(A)$ transforms to $x' = x'(A)$ since $x' = dA/dx = F^{-1}$. To transform the resulting curve (shown dashed on the intermediate plane Ax') to the Ax plane, we must integrate it with respect to A, i.e., keeping OA unaltered, transform in accordance with the rules of graphical integration from the x'- to the x-axis.

The corresponding constructions are shown in Fig. 7. Since integration is carried out for arbitrary initial conditions we can, in addition to the $x = x(A)$ curve, which is also a graph of $A = A(x)$, obtain a number of other curves differing from each other only by the shift along the x-axis. Curves 1 and 2 of Fig. 8 are shifted relative to each other along the x-axis and are shown in the first quadrant, while the basic function $k = k(A)$ is shown in the second. The required function $k = k(x)$ obtained for curves 1 and 2 by the above method is shown in the fourth quadrant.

For more complicated inhomogeneous media, for example, for

$$k(A) = b\,e^{aA}, \tag{3.6}$$

we can still use the results obtained for the simple problem of Eq. (3.1). In fact, Eq. (3.1) can be written in the form $k(A) = be^{aA}$, where $a = -1$ and $b = 1$. Therefore, if we construct a graph of $A = A(x)$ with, for example, $a = -1$, $b = 2$, then, in accordance with recommendations of Section 3d of [1], it is sufficient to transform the graphs for Eq. (3.1) only slightly. All that is necessary is to compress the curves of the first quadrant in Fig. 8 along the horizontal direction by a factor of 2 and retain the scale along the vertical direction. Consequently, in the fourth quadrant we must compress the corresponding curves along the horizontal axis by a factor of 2 and expand them by the same factor along the vertical axis. Curves 1 and 2 which are obtained as a result of this transformation are shown in Fig. 9a.

Similarly, if the basic function is

$$k(A) = 2e^{-0.5A}, \tag{3.7}$$

i.e., if $a = -0.5$, $b = 2$, the graphs of $A = A(x)$ and $k = k(x)$ of Fig. 8 must be compressed along the horizontal axis and expanded along the vertical axis (by a factor of two). The resulting curves are shown in Fig. 9b.

If we now wish to obtain the resultant field $\psi(x)$ in the inhomogeneous medium, we must determine the particular solution $Y_1 \exp i\Psi_1$ of Eq. (0.1*) corresponding to Eq. (3.1). This means that we must first plot a graph of $\varphi = \varphi(A)$ on the $AO\varphi$ plane found by integrating the curve $\varphi' = d\varphi(A)/dA$ obtained in accordance with Eq. (2.9*):

$$\varphi' = \left\{\frac{p(A)}{A} + \left[\frac{k(A)}{F(A)}\right]^2\right\}^{1/2} = \left[\frac{1}{(Fe^A)^2} - \frac{1}{A}\right]^{1/2}. \tag{3.8}$$

The initial conditions are taken into account in Eq. (1.1*) by suitably choosing the integration constants C_1 and C_2, and the equations of (2.11*) are used to calculate the parameters

$$M(A) = \left[\frac{1}{(Fe^A)^2} - \frac{1}{A}\right]^{1/2} - \frac{1}{Fe^A}, \qquad N(A) = \frac{1}{A} - 1.$$

Equation (2.12*) is used after the change of variable $A \to x$, shown in Fig. 10, to determine the functions $Y_1 = Y_1(x)$ and $\Psi_1 = \Psi_1(x)$.

The resultant field in the inhomogeneous medium of Eq. (3.1) is described by Eq. (1.1*), and includes in addition to $Y_1(x)$ and $\Psi_1(x)$ the further functions $Y_2(x)$ and $\Psi_2(x)$. The latter functions are found by constructing with the aid of Eq. (2.14*) the graphs of $R_1 = R_1(A)$ and $R_2 = R_2(A)$, followed by integration of these graphs with respect to A, and the determination of $J_1(A)$ and $J_2(A)$. The functions $Y_2 = Y_2(A)$ and $\psi_2 = \psi_2(A)$ calculated from Eq. (2.13*) after the change of variable $A \to x$ are shown in Fig. 10.

The solution of the problem defined by (3.1) is thus given by the graphs of $Y_1 = Y_1(x)$, $\Psi_1 = \Psi_1(x)$, $Y_2 = Y_2(x)$ and $\Psi_2 = \Psi_2(x)$, shown in Fig. 10, which describe the general solution of Eq. (0.1*)

$$\psi(x) = Y_1 e^{i\Psi_1}\,[C_1 + C_2 Y_2 e^{-i\Psi_2}].$$

This is the resultant field in the inhomogeneous medium with $k = k(x)$ characterized by curve 1 of Fig. 8.

It was shown in [1] that the wave-propagation problem can also be solved by the above asymptotic method in a purely analytic form. The necessary transformations (change of variable and integration) are then carried out numerically on a computer and not graphically. The use of the graphical constructions in the present paper was dictated by the desire to exhibit more directly the general features of this method of solution.

In conclusion, we note the following facts:

1. The process of solution of the above problems by the asymptotic method does not involve any assumptions such as, for example, those

employed in the short-wave or long-wave approximations. The accuracy of the method is governed only by errors introduced during the intermediate transformations. The solution of the wave-propagation problem for an arbitrary inhomogeneous medium can therefore be carried out by this method to any given accuracy not only for the field $\psi(x)$, but also for the partial waves $\alpha(x)$ and $\beta(x)$ [4], and the internal reflection coefficient $R(x) = \beta(x)\alpha^{-1}(x)$.

2. The above explains the complexity of the method which is, in fact, an accurate numerical method. It is therefore more difficult to use it to solve the propagation problem for inhomogeneous media than certain other special methods which are effective only for a particular class of media (weakly inhomogeneous, finely stratified with a sinusoidal dependence of wave number on distance, and so on). However, in the case of more complicated media characterized, for example, by an oscillating variation of the wave number with distance (including variable amplitude and period), the present method has definite advantages as compared with other existing methods.

3. In view of its generality, the asymptotic method is useful in finding the solution for inhomogeneous media $k = k(x)$ for which existing methods have not resulted in a solution. Since the three stages of solution in [1] are the same for all types of medium, we can use the same program to solve the propagation problem for different classes of complicated inhomogeneous media using digital computers, and this also simplifies the corresponding work for analog computers.

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